# Mode Localization Phenomena in Large Space Structures

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**The possibility of localization or confinement of vibratory modes in large space structures is investigated theoretically and numerically. These structures belong to a class of periodic structures that have recently been shown to be sensitive to periodicity-breaking disorder or imperfections. When localization occurs, the modal amplitude of a global mode becomes confined to a local region of the structure, with serious implications for the control problem. The results of this study indicate that mode localization is most likely to occur in structures consisting of a large number of weakly coupled substructures. Certain large space structures with high modal densities fall in this category, and it is therefore important to include the effect of structural imperfections and disorder when designing control systems for shape or directional control of such structures.**

## Nomenclature

$[I]$ = identity matrix
$K, K_h, K_c$ = stiffness coefficients defined in Fig. 1
$K_w, K_u, K_\theta$ = stiffness coefficients defined in Fig. 10
$[K]$ = stiffness matrix
$[k]$ = stiffness perturbation matrix
$[M]$ = mass matrix
$[m]$ = mass perturbation matrix
$m$ = mass defined in Fig. 1
$n$ = wave number
$N$ = number of substructures
$p$ = number of degrees of freedom per substructure
$q_i$ = generalized coordinate
$\bar{q}_i$ = amplitude of $q_i$
$t$ = time
$u, v, w$ = displacements in $x, y, z$ directions
$u_i$ = substructure coordinates
$\delta_{ij}$ = Kronecker delta
$\epsilon$ = imperfection parameter, $\ll 1$
$\theta$ = torsional displacement
$\lambda_n$ = $\omega_n^2$ = eigenvalue
$\nu_m, \psi_m$ = $m$th order perturbation terms, Eq. (19)
$\sigma_n$ = phase angle, Eq. (10)
$\omega$ = frequency in rad/s
$\bar{\omega}$ = $\omega/\omega_0$ = nondimensional frequency
$\{\phi\}_n$ = eigenvector
$(\tilde{\ })$ = perturbed quantity

## Introduction

LARGE space structures have received considerable attention in the dynamics and control literature recently. The proposed applications are many, spanning diverse areas such as solar energy collectors, solar sails, large astronomical telescopes, communication antennas, and space station structures.[1,2] Because structural weight must be kept to a minimum, the structures are relatively flexible and require sophisticated control systems if accurate shape or directional control must be maintained.

A typical feature of these designs is the fact that the nominal structure has cyclic symmetry, or some form of periodicity. Often, the structure is an assembly of a large number of relatively simple substructures, each consisting of only a few beam or truss elements. The analysis of such a structure can be simplified considerably by taking advantage of its periodicity, and a number of investigations have been devoted to this problem.[3-7] The subject of vibration and wave propagation through periodic structures is hardly new however, since it also played an important role in the theory of solids; see, for example, the classical work by Brillouin.[8]

Recently, periodic structures have been shown to be sensitive to certain types of periodicity-breaking disorder or imperfections, resulting in a phenomenon known as mode localization or confinement. Structures consisting of a large number of weakly coupled substructures are especially susceptible. Certain large space structures with high modal densities belong to this class, and it is therefore important to consider structural imperfections when designing control systems for shape or directional control of such structures. In fact, it is possible that the concept of robust controls needs to be redefined for structures where mode localization occurs.

This paper addresses some of the fundamental questions related to the mode localization problem. The approach is novel in that it draws together ideas and results from several different fields, notably solid-state physics, solid mechanics, and dynamics, where the same phenomenon occurs under different names. Although the study is directed toward large space structures, the results are believed applicable to a wide class of periodic structures.

## Imperfection Sensitivity and Mode Localization

The importance of imperfections in stability problems is now well known. In fact, the theory of elastic stability has provided some of the most illuminating examples of systems that are sensitive to imperfections. The study of imperfection sensitivity in connection with static stability problems received considerable attention after the pioneering work of Koiter,[9] and contributions have been made by many researchers in engineering and applied mathematics, of which Refs. 10–12 are representative. The application of Tom's catastrophe theory to this problem has also received much attention recently.[13]

The effect of imperfections in *dynamic* stability and control problems is not as well understood, especially if nonconservative forces (such as in the flutter problem) are present. For certain periodic structures, e.g., a cascade of identical airfoils, the introduction of imperfections or "disorder" can actually be stabilizing and hence beneficial. In a recent paper,[14] it was

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shown that the stabilization mechanism is closely connected to the phenomenon of mode localization or confinement, which often occurs in periodic structures when the periodicity is broken by disorder or imperfections.

Periodic structures have some remarkable dynamical properties, which have been discovered and rediscovered in several diverse areas of engineering and solid-state physics. It is well known that the normal modes of vibration of such structures are extended; that is, the standing wave comprising the mode extends throughout the structure. When the periodicity of the structure is broken, however, it is found that certain modes may become highly *localized* to only a few of the substructures.

In solid-state physics, this phenomenon has attracted considerable attention after it was shown by Anderson and Mott to explain many of the transport properties of disordered solids; see, for example, Refs. 15 and 16. If the state vectors or eigenmodes of the Schrödinger wave equation governing the motion of electrons in a solid should become localized, metallic conduction would become impossible. Thus, an "ordered" solid (i.e., one with a regular crystal lattice) may change abruptly from a metallic conductor to a semiconductor once this order is destroyed by imperfections or impurities, or when the solid becomes amorphous. The importance of this discovery can be gaged by noting that Anderson and Mott shared the 1977 Nobel Prize in physics for their work in this area.

In the context of structural dynamics, mode localization has been discussed by Hodges[17] using relatively simple models, and also in a very recent study by Pierre.[19] The present author has shown that the same phenomenon is responsible for the large scatter (often exceeding 100%) observed in the individual blade vibration amplitudes of turbine and compressor rotors.[14,18] This is not a desirable situation from a fatigue standpoint, and the fatigue life of high-amplitude blades can therefore be seriously reduced by the mode localization phenomenon. In the engineering literature, this is known as the "mistuned" rotor problem, since the imperfections result in small variations in the natural frequencies of the individual blades.

In some periodic structures, such as the bladed compressor or turbine disks described, significant mode localization can be caused by structural irregularities within the range of manufacturing tolerances. If this is possible in large space structures, serious operational difficulties are likely to occur if accurate shape control is required. Also, because of the "spill-over" problem in all real control systems, stability and controllability issues would have to be addressed if localized modes occur.

A serious consequence of localization is that it destroys the regular features of the mode, such as the regular spacing of nodal points and lines, and the sinusoidal amplitude modulation. Both the amplitudes and phase relationships of sensor signals would become radically different from their expected values. Any control system designed based on the "tuned" modes of the ideal structure would find itself attempting to control a structure whose dynamics is essentially unknown. Preliminary results obtained by the author indicate that the identification problem may also become a major challenge, making it difficult to apply the concepts of adaptive controls.

## Localization of Normal Modes

Before discussing the possibility of mode localization in large space structures, it is worth mentioning that several definitions of localization have been proposed (for a readable discussion, see Ref. 20). Since the degree of localization is a function of the amount of "disorder" or structural imperfections present, a gray area must necessarily exist between what one considers extended vs localized modes. In very large periodic one-dimensional structures, almost all of the modes are localized, regardless of the amount of disorder, provided one looks on a sufficiently large scale. In two dimensions, the situation is believed to be similar, whereas in three dimensions, it is believed that a finite threshold of disorder is needed in order to localize all modes.[17] Incidentally, a complete "theory of localization" does not exist for two- and three-dimensional structures, and in the words of Anderson, "... one has to resort to the indignity of numerical experiments to settle even the simplest questions about it."[16]

For large space structures, it may be useful to interpret localization in terms of what happens to waves propagating along the structure. When structural irregularities are introduced, multiple scattering (or reflections, if you wish) at the substructure junctions may lead to localization. As the length of the structure becomes infinite, localization is indicated by the structure becoming perfectly reflecting. Of course, for a practical structure, the number of substructures may be large but never infinite; hence, the mode confinement may be less spectacular. The numerical results obtained in this study indicate that, surprisingly often, the confinement is indeed spectacular.

### Single-Degree-of-Freedom Substructures

The essential features of mode localization in periodic structures are best demonstrated using a simple model. Figure 1 illustrates such a generic model for a class of structures with cyclic summary, e.g., disk antennas. Each substructure is modeled with one degree of freedom; however, this degree of freedom may be regarded as any one of the possibly infinite number of degrees of freedom of the actual substructure. In the case of an antenna, this could be one of the beam bending modes of the ($N$) antenna ribs. The structural coupling between the substructures is represented by the springs $K_c$; this parameter (suitably normalized) is of fundamental importance in the problem. The spring $K_h$ represents the hub or base attachment stiffness and provides a convenient means for extending the structural coupling beyond next neighbors. Note that the structure is essentially one-dimensional from a dynamics standpoint, since disturbances travel in only one dimension (the circumferential direction).

### Normal Modes of Perfect Structure

If one considers a structure of axial symmetry, such as a disk antenna, then two types of periodicities are present. First, a perfect structure assembled from identical substructures is cyclic or periodic in the circumferential direction with a "wavelength" equal to the substructure spacing. In addition, a global periodicity exists in the sense that the first substructure is next to the last substructure. An important consequence of this cyclic structure is that certain rows and columns of the system mass and stiffness matrices are cyclic permutations of each other. This follows from the fact that, for such a structure assembled from $N$ *identical* substructures with $p$ degrees of freedom each (in the assembly), the system matrices must

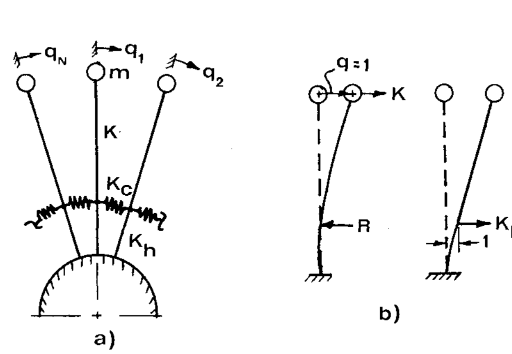


**Fig. 1 Simple model of a structure with cyclic symmetry. Structure is periodic in the circumferential direction. a) Reference substructure, $n = 1$, with adjoining substructures $n = 2$ and $n = N$. b) Substructure (beam) stiffness coefficients.**

be left unchanged by a shift permutation of the form

$$q_i \to q_{i+np} \qquad n = 1, 2, \ldots \tag{1}$$

where the periodicity condition

$$q_{i+pN} = q_i \tag{2}$$

is also implied due to the "circular periodicity" of the structure. Thus, the $(n+p)$th row of any of the assembled system matrices must be a cyclic permutation of the $n$th row, consisting of a circular shift right by $p$ positions. Similarly, the $(m+p)$th column is a circular shift down by $p$ positions of the $m$th column.

The previous example illustrates what one may call a "closed" periodic structure; it has no beginning or end in the sense that the first and last substructures are joined. In an "open" periodic structure, then, substructures number 1 and $N$ would not be joined, and the periodicity condition given by Eq. (2) would not hold. Waves traveling along an open periodic structure would suffer successive reflections, which would depend on the boundary conditions present at the first and last substructures. The same concepts can obviously be extended to two-dimensional structures, but not to three-dimensional ones.

In the case of the single-degree-of-freedom (per substructure) model shown in Fig. 1, the mass and stiffness matrices are

$$[M] = m[I] \tag{3}$$

$$[K] = K\left([I] - K[K]_h^{-1}\right) \tag{4}$$

Here $[I]$ is the $N \times N$ identity matrix, and

$$[K]_h = (K + K_h)[I] + [K]_c \tag{5}$$

$$[K]_c = K_c \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & 0 & \cdots & 0 & 0 \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ -1 & 0 & \cdot & \cdot & 0 & -1 & 2 \end{bmatrix} \tag{6}$$

It should be noted that $[K]$ is a fully populated stiffness matrix because of $[K]_h^{-1}$ in Eq. (4); thus, structural coupling extends beyond adjacent substructures.

If we assume simple harmonic motion, $\{q\} = \{\bar{q}\}\exp(i\omega t)$, we obtain an eigenvalue problem of the general form

$$[K]\{\bar{q}\} = \omega^2[M]\{\bar{q}\} \tag{7}$$

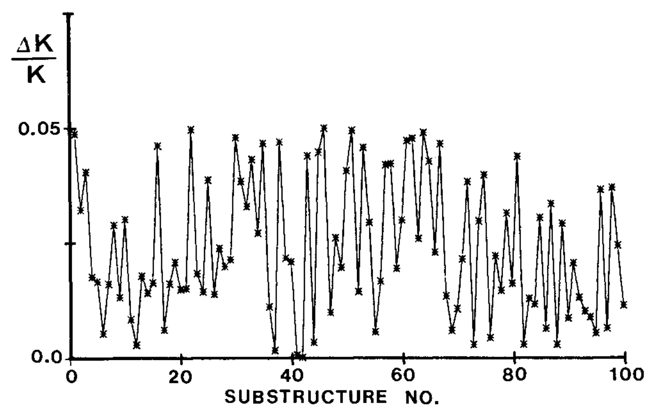


**Fig. 2 Random imperfections in $K$, with uniform probability in interval [0, 0.05].**

In this example, the system matrices are circulant: each row is a circular shift right by one position of the previous row. Circulant matrices have simple and well-defined spectral properties,[21] which actually allow us to write down the solution to the dynamics eigenvalue problem [Eq. (7)], as

$$\omega^2 = \omega_n^2 = \frac{\{\phi\}_n^*[K]\{\phi\}_n}{\{\phi\}_n^*[M]\{\phi\}_n} = \omega_0^2\left[\frac{\bar{K}_h + 2\bar{K}_c(1-\cos\sigma_n)}{1 + \bar{K}_h + 2\bar{K}_c(1-\cos\sigma_n)}\right] \tag{8}$$

$$\{\bar{q}\}^T = \{\phi\}_n^T = \{\rho_n^0\ \rho_n^1\ \rho_n^2 \cdots \rho_n^{N-1}\}/\sqrt{N} \tag{9}$$

Asterisks denote Hermitian conjugate of a matrix, and

$$\bar{K}_h = K_h/K, \qquad \bar{K}_c = K_c/K$$
$$\omega_0^2 = K/m, \qquad \rho_n = \exp(i\sigma_n)$$
$$\sigma_n = 2\pi n/N, \qquad n = 0, 1, 2, \ldots, N-1 \tag{10}$$

Here, the $\rho_n^m$ occurring in the eigenvectors are simply the $N$th roots of unity $\rho_n$ raised to a power $m$. The orthonormal eigenvectors $\{\phi\}_n$ can also be written in terms of real vectors as ($N$ even)

$$\{\phi_c\}_n = \frac{1}{\sqrt{N}}\begin{Bmatrix} 1 \\ \cos\sigma_n \\ \cos 2\sigma_n \\ \vdots \\ \cos(N-1)\sigma_n \end{Bmatrix} \qquad (n = 0, 1, 2, \ldots, N/2) \tag{11a}$$

$$\{\phi_s\}_n = \frac{1}{\sqrt{N}}\begin{Bmatrix} 0 \\ \sin\sigma_n \\ \sin 2\sigma_n \\ \vdots \\ \sin(N-1)\sigma_n \end{Bmatrix} \qquad (n = 1, 2, 3, \ldots, N/2 - 1) \tag{11b}$$

If $N$ is odd, the range of $n$ in Eq. (11) extends up to $(N-1)/2$. Physically, the integer $n$ in Eq. (11) corresponds to the number of nodal diameters associated with the natural mode. Except for $n = 0$ or $N/2$, multiple eigenvalues occur, since both $\{\phi_c\}_n$ and $\{\phi_s\}_n$ are eigenvectors corresponding to the eigenvalue $\omega_n^2$. The vectors are always linearly independent.

The important point to note from the foregoing results is that the natural modes of the perfect system, Eq. (11), are extended, regardless of the order of the mode. The sinusoidal amplitude modulation evident in the standing waves representing the modes is a well-known result from wave-propagation theory in periodic structures.[8] These standing waves can be considered a superposition of two identical waves traveling in opposite directions along the structure. A traveling wave is conveniently written in terms of the complex eigenvectors, Eq. (9), as

$$\{q\}_n = \mathrm{Re}\left(\{\phi\}_n e^{i\omega_n t}\right) = \frac{1}{\sqrt{N}}\begin{Bmatrix} \cos\omega_n t \\ \cos(\omega_n t + \sigma_n) \\ \cos(\omega_n t + 2\sigma_n) \\ \vdots \\ \cos(\omega_n t + (N-1)\sigma_n) \end{Bmatrix} \tag{12}$$

Where no confusion can arise, the coordinate vector will be taken as complex, with the real part understood to represent the physical coordinates.

### Effect of Imperfections and Disorder

No structure can be manufactured precisely identical to any other structure, and this must also be true for the substructures making up a large space structure. In order to illustrate the drastic effect such periodicity-breaking imperfections can have on the mode shapes, random imperfections were introduced into the substructures by altering the stiffness $K$ in Fig. 1 by an amount $\Delta K$, where $\Delta K/K$ is a random variable with a uniform probability density function in the interval [0, 0.05]. The specific distribution of imperfections used in generating the results in this paper is shown in Fig. 2, for the [0, +5%] case. This choice simulates a relatively narrow quality control band, with limits of +5%, −0%, on $K$.

For coupling strength $\bar{K}_c = K_c/K = 1$ and $K_h/K = 10$, surprisingly strong mode localizations occur, as shown by the solid lines in Figs. 3–5. Three typical modes are shown, corresponding to 2, 5, and 25 nodal diameters. The overall structure has 100 substructures, with the first and last structures connected to simulate a circular structure. As can be seen, the vibrational amplitude is confined to very few adjacent substructures, with the other substructures remaining virtually stationary. The locations of the peaks cannot be anticipated from a cursory inspection of the actual distribution of imperfections, Fig. 2, and different modes typically localize at *different* substructures. Note that the recognizable features of the original mode (dashed lines), such as the number of nodal lines, are totally obliterated. The fact that the nodes are now in new and unpredictable locations is obviously a serious problem from a control standpoint.

These relatively simple results demonstrate dramatically the phenomenon of mode localization or confinement in the context of structural vibrations. A qualitative explanation of localization in one-dimensional structures of the type shown in Fig. 1 can be found in Ref. 14; see also Ref. 17. Based on theoretical arguments, one would expect that structures with relatively weak coupling (modeled by $K_c$ in Fig. 1) between individual substructures should be most susceptible to mode localization. Furthermore, it is the ratio of imperfection "amplitude" to interstructure coupling strength $\bar{K}_c$ that matters, rather than the imperfection amplitude itself.

### Multi-Degree-of-Freedom Substructures

The situation is qualitatively similar when each substructure is modeled with more than one degree of freedom. The vector of generalized coordinates for the *assembled* structure can be partitioned as follows:

$$\{q\}^T = \{\{q\}_1^T | \{q\}_2^T | \cdots | \{q\}_N^T\} \tag{13}$$

where $\{q\}_i$; $i = 1, 2, \ldots, N$ are the generalized coordinate vectors for the individual substructures. Before assembly, each of the $N$ identical substructures is modeled by a set of $m$ local coordinates $\{u\}$, which may be partitioned into $m - 2n_c$ internal coordinates $\{u_i\}$ and $2n_c$ coupling (boundary) coordinates $\{u_c\}$. The coupling coordinates may be further partitioned into $n_c$ coordinates at each of the left and right boundaries. In the assembly, then, each periodic element is represented by $p = m - n_c$ coordinates $\{q\}_i$, of which $n_c$ are "coupling coordinates" and $p - n_c$ are interior coordinates.

### Normal Modes of Perfect Structure

For a perfect structure, one in which each substructure is *identical*, the shift permutation on the coordinates given by Eqs. (1) and (2) must leave the global (system) matrices unaltered. It follows then that $[K]$ is of the form

$$[K] = \begin{bmatrix} [K]_{11}[K]_{12}[K]_{13} & \ldots & [K]_{13}^T[K]_{12}^T \\ [K]_{12}^T[K]_{11}[K]_{12} & \ldots & [K]_{14}^T[K]_{13}^T \\ \cdots\cdots & \ldots & \cdots \\ [K]_{12}[K]_{13}[K]_{14} & \ldots & [K]_{12}^T[K]_{11} \end{bmatrix} \tag{14}$$

Here, the $p \times p$ submatrices $[K]_{1j}$ represent the structural coupling between the first and $j$th substructures; $j = 2, 3, \ldots, N/2$ [or $(N-1)/2$ if $N$ is odd], and the circular symmetry requires that $[K]_{1(N+2-j)} = [K]_{1j}^T$, $j = 2, 3 \ldots$. The mass matrix $[M]$ has the same overall structure. If $p = 1$, one recovers the circulant matrices of the one-degree-of-freedom per substructure model as previously discussed. By analogy, it seems reasonable to call matrices of the form given by Eq. (14) "block-circulant." Carrying the analogy a little further, one would expect from Eqs. (7–10) that the eigenvector for the multi-degree-of-freedom case should be of the form

$$\{\bar{q}\} = \begin{Bmatrix} \{\bar{q}\}_1^n \\ \{\bar{q}\}_1^n e^{i\sigma_n} \\ \{\bar{q}\}_1^n e^{2i\sigma_n} \\ \vdots \\ \{\bar{q}\}_1^n e^{(N-1)i\sigma_n} \end{Bmatrix} \tag{15}$$

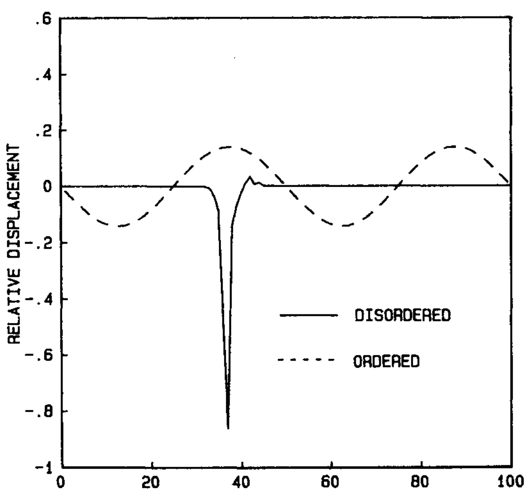


**Fig. 3 Severe localization of a two-nodal diameter mode due to random imperfection/disorder in $K$: $K_h/K = 10$; $K_c/K = 1$; $N = 100$; 5% random variation in $K$.**

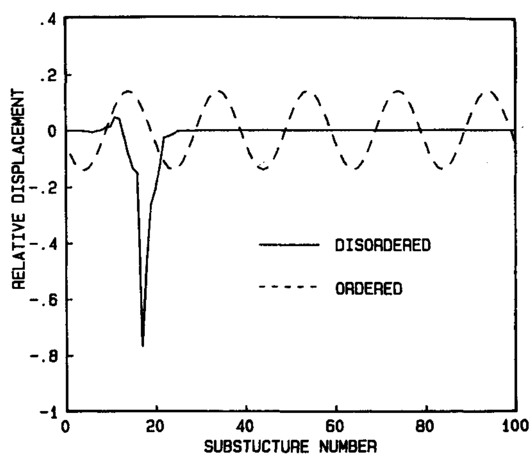


**Fig. 4 Severe localization of a five-nodal diameter mode due to random imperfection/disorder in $K$: $K_h/K = 10$; $K_c/K = 1$; $N = 100$; 5% random variation in $K$.**

where $\sigma_n$ is given by Eq. (10) and $\{\bar{q}\}_1^n$ is a $p$-dimensional vector. The subscript indicates that the first substructure has been chosen as the reference structure, and the superscript that the vector depends on $\sigma_n$.

The validity of Eq. (15) is readily verified by substituting into the eigenvalue problem, Eq. (7), and making use of the block-circulant properties of $[K]$ and $[M]$ and the restriction on $\sigma_n$ given by Eq. (10). The result is a set of $N$ uncoupled eigenvalue problems of order $p$:

$$\left([K(\sigma_n)]_R - \omega^2[M(\sigma_n)]_R\right)\{\bar{q}\}_1^n = 0, \qquad n = 0,1,2,\ldots,N-1 \quad (16)$$

The reduced stiffness and mass matrices can be written in terms of the submatrices $[K]_{1j}$ and $[M]_{1j}$ and the phase angle $\sigma_n$:

$$[K(\sigma_n)]_R = [K]_{11} + \sum_{j=2}^{N/2}\left\{\left([K]_{1j} + [K]_{1j}^T\right)\cos\sigma_n + i\left([K]_{1j} - [K]_{1j}^T\right)\sin\sigma_n\right\} \quad (17)$$

with an analogous expression for $[M(\sigma_n)]_R$. If $N$ is odd, the sum in Eq. (17) extends up to $(N-1)/2$. The reduced stiffness and mass matrices are thus Hermitian, which guarantees that the eigenvalues $\omega^2$ are real. It suffices to solve Eq. (16) for $n = 0,1,2,\ldots,N/2$ [or $(N-1)/2$ if $N$ is odd], since $n = j$ and $n = N - j$ yield the same eigenvalues but with complex conjugate eigenvectors. For $\sigma = 0$ and $\pi$, the matrices are real and the eigenvalues simple. The foregoing reduction procedure is well known for structures with cyclic symmetry, although different authors often use different arguments to arrive at Eq. (16).

### Effect of Imperfections and Disorder

When the cyclic symmetry of the structure is destroyed by imperfections, the reduction procedure fails and the size of the eigenvalue problem that must be solved increases from $p$ to $pN$. Numerical studies of the effect of imperfections in large structures could therefore be expensive and would also face the numerical difficulties associated with very large eigenvalue problems.

If the magnitude of the imperfections can be assumed small, such as may be expected from manufacturing tolerances, it is natural to use perturbation methods. In this approach, the imperfect or disordered structure is considered a perturbed periodic structure, where the perturbations are small variations in the structural and inertial operators. For structures modeled with a finite number of degrees of freedom, the perturbed mass and stiffness matrices $[M]$ and $[K]$ are written as

$$[\tilde{M}] = [M] + \epsilon[m]$$
$$[\tilde{K}] = [K] + \epsilon[k] \quad (18)$$

where $\epsilon \ll 1$, and the perturbation matrices $[m]$ and $[k]$ are of the same order as the unperturbed matrices.

In the present study, the perturbation scheme is not used in actual numerical simulations, but rather as a theoretical framework in which to study the sensitivity of the eigensolution to the parameter $\epsilon$ and the perturbation matrices. In the limit $\epsilon \to 0$, the perturbation solutions are asymptotic to the exact solution of the perfect (periodic) system, corresponding to the limiting value $\epsilon = 0$. The eigensolution of the perturbed system is sought as asymptotic expansion involving $\epsilon$. If the $n$th eigenvalue $\lambda_n = \omega_n^2$ is simple, the corresponding perturbed eigenvalue and eigenvector can be written as expansions of the form

$$\tilde{\lambda}_n = \lambda_n + \epsilon\nu_1 + \epsilon^2\nu_2 + \ldots$$
$$\{\tilde{\phi}\}_n = \{\phi\}_n + \epsilon\{\psi\}_1 + \epsilon^2\{\psi\}_2 + \ldots \quad (19)$$

[For multiple eigenvalues, the procedure must be modified slightly (see Ref. 22), but this is not important for the subsequent discussion.] The unknown correction terms $\nu_m$ and $\{\psi\}_m$ can be determined by solving a sequence of nonhomogeneous problems for terms of successive orders: $1, \epsilon, \epsilon^2, \ldots$. The procedure and results are well known; see, for example, the classical treatment of perturbed eigenvalue problems given in Courant and Hilbert.[22] In Ref. 18, perturbation formulas were obtained for "almost periodic" structures in the presence of nonconservative forces, e.g., the flutter problem, where the matrices may be complex and non-Hermitian. For the special case of free vibrations of a stiffness-perturbed structure, the perturbation solution, Eq. (19), is given to second order in $\epsilon$ by

$$\nu_1 = \kappa_{nn} \quad (20)$$

$$\nu_2 = \sum_{m=1}^{pN}{}' \frac{1}{(\lambda_n - \lambda_m)}\kappa_{mn}^2 \quad (21)$$

$$\{\psi\}_i = \sum_{m=1}^{pN} c_{im}\{\phi\}_m \quad (22)$$

where the prime on the summation sign indicates that the term $m = n$ is to be omitted. The eigenvectors are assumed real and normalized with respect to the mass matrix; i.e.,

$$\{\phi\}_i^T[M]\{\phi\}_j = \delta_{ij} \quad (23)$$

and the coefficients in Eq. (22) are as follows:

$$\kappa_{mn} = \{\phi\}_m^T[k]\{\phi\}_n \quad (24a)$$

$$c_{1m} = \kappa_{mn}/(\lambda_n - \lambda_m) \quad \text{for } m \neq n$$
$$= 0 \quad \text{for } m = n \quad (24b)$$

$$c_{2m} = \frac{1}{(\lambda_n - \lambda_m)}\left\{\sum_{j=1}^{pN}\kappa_{mj}c_{1j} - \nu_1 c_{1m}\right\} \quad \text{for } m \neq n$$
$$= -\frac{1}{2}\sum_{j=1}^{pN} c_{1j}^2 \quad \text{for } m = n \quad (24c)$$

Similar formulas can also be obtained for perturbations of $[M]$; see Ref. 18 (but note that $\lambda = 1/\omega^2$ is taken as the eigenvalue in Ref. 18; hence, the role of mass and stiffness perturbations are reversed from those in the present paper).

The perturbation formulas provide important insight into the basic mechanism responsible for mode localization. Note first that one important effect of imperfections is to couple modes that were orthogonal to each other in the perfect structure. The strength of this coupling, which to first order in $\epsilon$ is given by the coefficients $c_{1m}$, depends on two factors: 1) the matrix product $\kappa_{mn}$, Eq. (24a), involving the eigenvectors and the perturbation matrix and 2) the closeness of the unperturbed eigenvalues.

Consider first the coupling constants $\kappa_{mn}$, Eq. (24a). They are obviously well-behaved, and without loss of generality the perturbation matrix $[k]$ can be chosen such that $\kappa_{mn} \leq 1$ for all $m$. In the general case of arbitrary imperfections, the $n$th perturbed eigenvector will contain components from all of the $m$ unperturbed vectors, unless $\kappa_{mn} = 0$ for some $m$. For random perturbations caused by manufacturing tolerances,

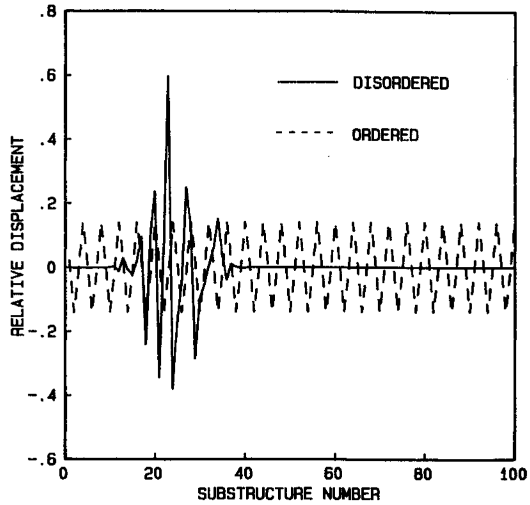


**Fig. 5 Severe localization of a 25-nodal diameter mode due to random imperfection/disorder in $K$: $K_h/K = 10$; $K_c/K = 1$; $N = 100$; 5% random variation in $K$.**

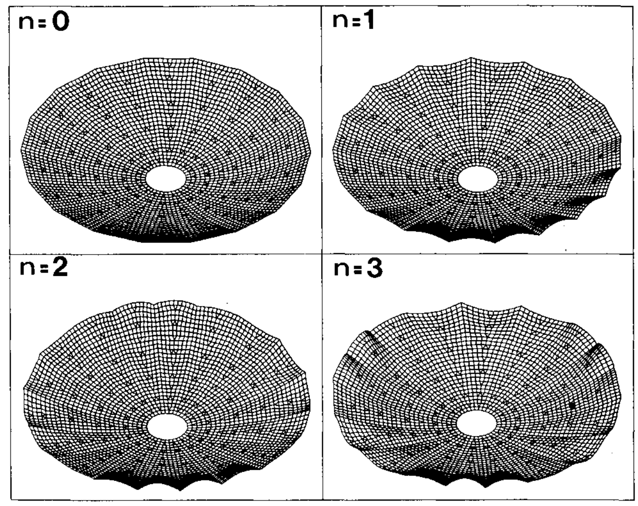


**Fig. 6 Typical normal modes of wrap-rib disk antenna studied in Ref. 7, corresponding to the first in-plane rib bending mode, group 2 in Table 1 (from Ref. 7).**

this would not be very likely, but for deterministic perturbations, it is always possible to choose the $[k]$ matrix such that $\kappa_{mn} = 0$ for certain values of $m$.

The second factor that determines the mixing of modes is the closeness of the unperturbed eigenvalues. Here is the first clue to how the localization mechanism works. If the structure has a large number of modes with frequencies close together, the differences $\lambda_n - \lambda_m$ in the denominators of the $c_{1m}$ coefficients, Eq. (24b), become small. The cumulative effect over $pN$ terms can then produce a correction term $\epsilon\{\psi\}_1$ to the eigenvector that is of order one rather than order $\epsilon$, as expected. Note that the corresponding correction term $\epsilon\nu_1 = \epsilon\kappa_{nn}$ to the frequency remains of order $\epsilon$, however. From Eq. (24c), it is apparent that the trouble gets worse with higher-order coefficients, since the second-order coefficients $c_{2m}$ have products of the form $(\lambda_n - \lambda_m)\cdot(\lambda_n - \lambda_j)$ in the denominators.

Thus, structures with high-modal densities are expected to be sensitive to imperfections, and a small change of order $\epsilon$ in a structural parameter may result in a much larger, say of order one, change in the natural modes. The behavior is similar to what has become known as a "catastrophe" in the recent applied mathematics literature; namely, small changes in a system parameter resulting in a large (and sometimes catastrophic) change in the system behavior.[13] The straightforward perturbation scheme may therefore break down for eigenvectors suffering strong localization. The ratio of successive terms may no longer be small, thus violating the underlying assumption of an asymptotic expansion. In this respect, the problem exhibits the features of singular perturbation problems.

The origin of the difficulty is not hard to find; it is, in fact, due to an inappropriate choice of the perturbation parameter. The perturbation series clearly suggests that it is not the imperfection/disorder strength per se that matters, but rather the ratio of this perturbation to the width of the frequency bands, $|\omega_n^2 - \omega_m^2|_{\max}$, corresponding to each modal group. Since the width of these frequency bands is a measure of the "interstructure coupling strength" (modeled by $K_c$ in Fig. 1), the appropriate disorder/imperfection parameter in this problem is $\epsilon/K_c$ or its equivalent. In fact, this is a well-known result of localization theory.[17]

A suitable limit process expansion can be based on letting $K_c(\epsilon) \to 0$ as $\epsilon \to 0$, so that the limit $\epsilon = 0$ corresponds to a perfect structure with zero coupling between the substructures. When the rate of $K_c(\epsilon) \to 0$ is specified, it defines a similarity parameter for the problem. One difficulty that must

**Table 1 Some of the natural frequencies (Hz) of the 15 m antenna (adapted from Ref. 7)**

| Mode no. | Circumferential wave no. $n$: 0 | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|---|
| 1 | 3.5613 | 3.5603 | 3.5588 | 3.5581 | 3.5585 | 3.5597 |
| 2 | 3.7590 | 3.7813 | 3.8440 | 3.9367 | 4.0465 | 4.1594 |
| 3 | 4.7815 | 4.7819 | 4.7832 | 4.7855 | 4.7888 | 4.7932 |
| 4 | 5.7357 | 5.7358 | 5.7360 | 5.7364 | 5.7368 | 5.7373 |
| 5 | 6.6103 | 6.6102 | 6.6101 | 6.6099 | 6.6095 | 6.6092 |
| 6 | 6.7259 | 6.7251 | 6.7227 | 6.7191 | 6.7146 | 6.7100 |
| 7 | 7.4559 | 7.4559 | 7.4559 | 7.4559 | 7.4560 | 7.4560 |
| 8 | 8.0672 | 8.0867 | 8.0853 | 8.0831 | 8.0805 | 8.0778 |
| 9 | 8.2621 | 8.2621 | 8.2622 | 8.2622 | 8.2622 | 8.2622 |
| 10 | 9.0221 | 9.0221 | 9.0219 | 9.0217 | 9.0215 | 9.0212 |
| 11 | 9.1701 | 9.1696 | 9.1683 | 9.1665 | 9.1645 | 9.1626 |
| 12 | 9.4938 | 9.4980 | 9.5102 | 9.5293 | 9.5534 | 9.5802 |
| 13 | 9.7454 | 9.7454 | 9.7455 | 9.7457 | 9.7459 | 9.7463 |
| 14 | 10.100 | 10.099 | 10.096 | 10.090 | 10.081 | 10.070 |
| 15 | 10.211 | 10.209 | 10.205 | 10.199 | 10.191 | 10.184 |
| 16 | 10.526 | 10.526 | 10.526 | 10.526 | 10.526 | 10.526 |

be overcome in this approach arises from the fact that each of the $p$ substructure frequencies now is $N$-fold degenerate, and the eigenvectors are only determined up to an arbitrary orthogonal transformation. Generally, the proper transformation must be determined by solving an eigenvalue problem (see Ref. 22); for an alternate approach, see Ref. 19.

All numerical results presented in this paper are based on an "exact" (numerical) solution of the full eigenvalue problem. The perturbation solution is only used in the theoretical discussion of localization.

## Numerical Example and Discussion

Based on the localization theory, one would expect that structures with relatively weak coupling between individual substructures should be most susceptible to mode localization. Since the width of the frequency band of a given substructure mode plotted vs circumferential wavenumber (or $\sigma_n$) is a measure of this coupling strength, structures with narrow frequency bands of high modal density belong to this class.

Examples of such structures are the large wrap-rib disk antennas studied in Ref. 7. Typical modes of the smaller (15 m diam) antenna are shown in Fig. 6 and Table 1. Note the high-modal density in addition to the very narrow frequency bands vs circumferential wavenumber $n$. Not surprisingly, the larger (55 m diam) antenna studied in Ref. 7 was observed to have an even higher modal density.

The frequency data in Table 1 was used to identify the structural stiffness terms in the simple model shown in Fig. 1. For the first cantilever bending mode of the ribs, corresponding to the second mode group in Table 1, the following quantities were used:

$$K_h/K = 2, \qquad K_c/K = 1.05$$

The 15 m antenna has 18 ribs; thus, each rib with the sectional wire mesh constitutes a substructure. In the present example however, 48 ribs and substructures were used, corresponding to the number of ribs in the 55 m antenna in Ref. 7.

Figures 7–9 illustrate the effect of random imperfections in the individual rib stiffness $K$, where $0 \le \Delta K/K \le 0.10$ with uniform probability. The imperfection distribution is the same as shown in Fig. 2 (for the first 48 substructures), but with twice the amplitude. Three modes are shown, corresponding to circumferential wavenumbers of $n = 0$, 1, and 18. The degree of localization is somewhat less spectacular than in the examples shown in Figs. 3–5, but the phenomenon is clearly present in all of the modes shown (and also in the remaining modes not shown). This apparent difference in the degree of localization is believed to be caused by the scaling effect when the number of substructures changed from $N = 100$ to $N = 48$, and is in agreement with theoretical expectations. Also, the effective coupling strength between the individual substructures is somewhat stronger than in the previous example.

A more elaborate model for the antenna would not be expected to alter these results to a significant degree. If the bending mode frequencies of the ribs are well separated, imperfections will not cause strong coupling between these modes, and each modal group will behave relatively independently. Normal mode localization would also be expected to occur in other types of large two-dimensional structures containing many weakly coupled substructures.

Figures 10 and 11 demonstrate the important role of the coupling strength between the substructures. Shown in Fig. 10 is a structure consisting of 24 beam elements with an additional rigid mass $M$ and springs $K_w$ and $K_\theta$ attached at the midpoint of each element or substructure. The beam element is a standard finite element based on Bernoulli-Euler beam theory, and the individual substructures are assumed to be rigidly attached to each other to form a closed periodic structure, i.e., with the 1st and 24th elements joined. Because the coupling between the substructures is strong, the modal frequency bands are wide and the structure is not expected to be sensitive to imperfections. This is clearly shown in Fig. 11, where a typical mode of the perfect structure is compared with the corresponding mode of a structure with 10% random variations in the substructure mass and stiffness matrices. No evidence of mode localization is present.

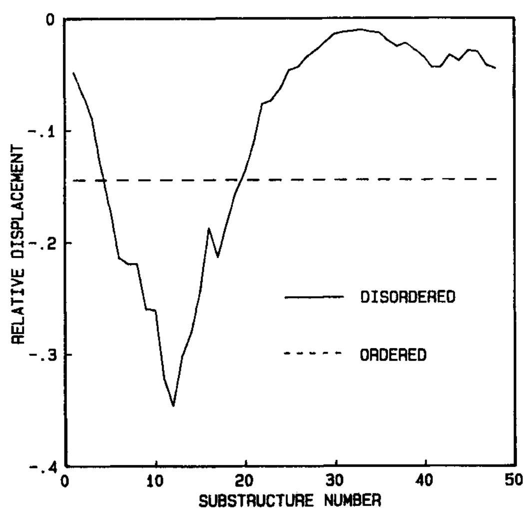


**Fig. 7 Evidence of localization of the zero-nodal diameter mode ($n = 0$) of a wrap-rib antenna, based on typical modal data from Ref. 7: $N = 48$; $K_h/K = 2$; $K_c/K = 1.05$; 10% random variation in $K$.**

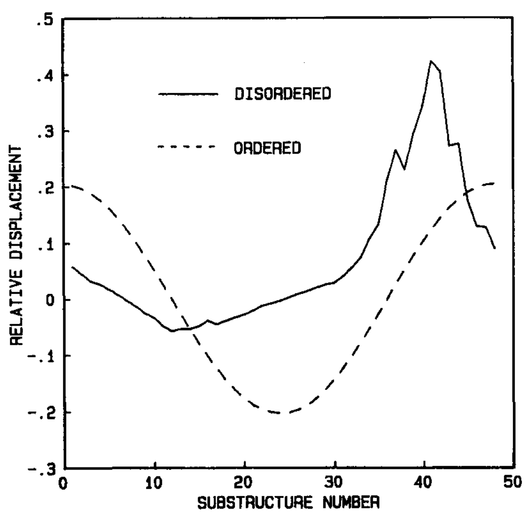


**Fig. 8 Evidence of localization of the one-nodal diameter mode ($n = 1$) of a wrap-rib antenna, based on typical modal data from Ref. 7: $N = 48$; $K_h/K = 2$; $K_c/K = 1.05$; 10% random variation in $K$.**

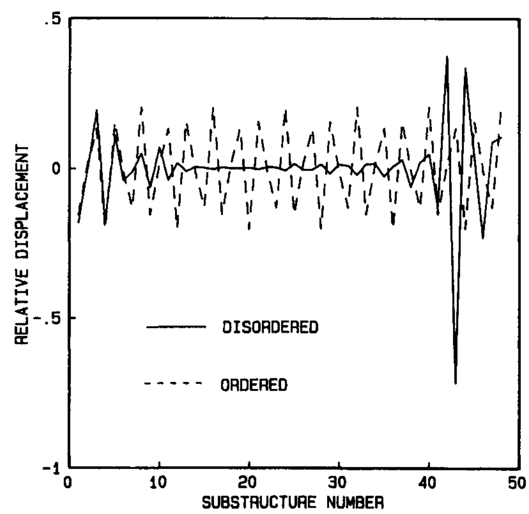


**Fig. 9 Evidence of localization of the 18-nodal diameter mode ($n = 18$) of a wrap-rib antenna, based on typical modal data from Ref. 7: $N = 48$; $K_h/K = 2$; $K_c/K = 1.05$; 10% random variation in $K$.**

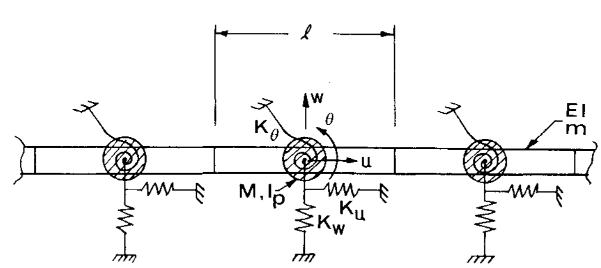


**Fig. 10 Periodic beam structure with cyclic symmetry.**

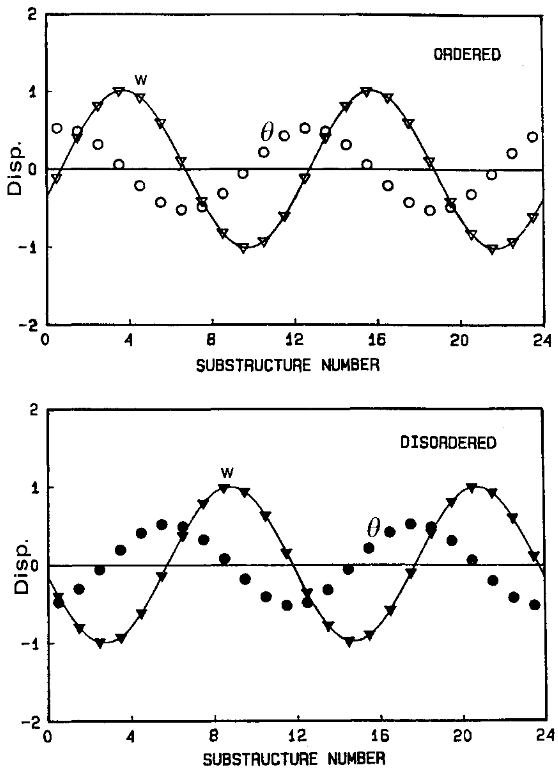


**Fig. 11 Absence of localization of typical mode ($n = 2$) in beam model (Fig. 1C), due to $\pm 10\%$ random variation in substructure mass and stiffness matrices: $N = 24$; $K_u\ell^3/EI = 0.1$; $K_w\ell^3/EI = 0.05$; $K_\theta\ell/EI = 0.1$; $M/m\ell = 10$; $I_p/m\ell^3 = 5$.**

## Conclusions

Periodic structures have been shown to be sensitive to periodicity-breaking disorder or imperfections, resulting in a phenomenon known as mode localization or confinement. Structures consisting of a large number of weakly coupled substructures are especially susceptible. Certain large space structures with high modal densities belong to this class, and it is therefore important to consider structural imperfections when designing control systems for shape or directional control of such structures.

The important parameter is the ratio of the imperfection strength to the effective coupling strength between the individual substructures. The results of this study suggest that for large periodic structures with weak interstructure coupling, significant mode localization can result from structural irregularities within normal manufacturing tolerances.

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